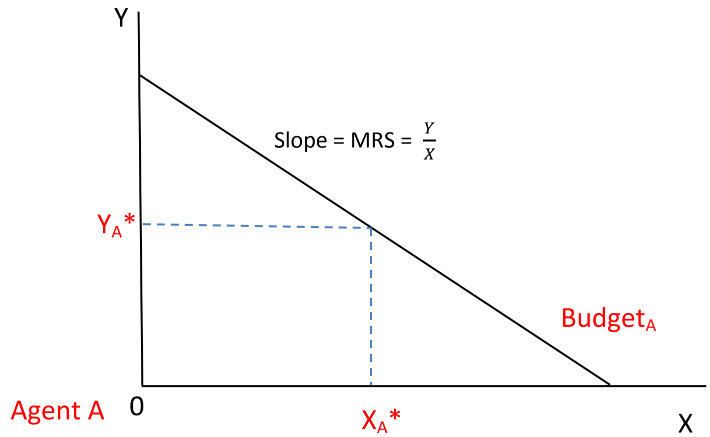
Consumer Budgets, Indifference Curves, and Utility Maximization[1]
Instructional Primer[2]

As rational, self-interested and utility maximizing economic agents, consumers seek to have the greatest level of overall satisfaction, but find themselves constrained by the possible choice of goods and services available to them and the budget with which they have to purchase these items. If you think about this, the never-completely-satisfied consumer wants the greatest level of utility they can afford and as such they will seek to consume that combination of goods and services that gives them the highest level of satisfaction, but that still lies on their budget curve… but just barely. This helps form the intuition behind the consumer's indifference curve being tangent to their budget curve.

Think about the consumer's budget in a system in which the consumer can only choose between two goods, X and Y. The consumer can spend the entire budget on X or the entire budget on Y. In this relation we think of the prices of goods as being exogenously[3] set for the markets, and the consumer's budget as being exogenously imposed on the consumer.



In this sense, the consumer must choose between the consumption (purchases) of good X and/or Y with their budget and we recognize that as the consumer moves up or down the budget curve from any given point they are substituting units of X for Y or Y for X in an effort to reach a maximum level of utility or satisfaction. As such, the slope of the budget curve is referred to as the Marginal Rate of Substitution (MRS), which is equal to $\frac{Y}{X}$.

While on the budget curve the consumer's budget is constant, so if the budget is $200, then the allocation the consumer will choose of goods X and Y are constrained by the budget amount while seeking the optimal allocation of X and Y to yield maximum consumer utility or satisfaction.

This relation falls out of a formation of consumer indifference curves and a budget curve. The indifference curve is that curve on which the consumer may obtain an allocation of goods (X,Y) such that the consumer has the same level of utility at any point on the curve, as such the consumer is indifferent and doesn't care which allocation is met, but that the utility level is the same.

Indifference curves have some basic rules:

1. They are downward sloping
2. Convex or quasi-convex

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[1] This primer is intended to present an abbreviated discussion of the included economic concepts and is not intended to be a full or complete representation of them or the underlying economic foundations from which they are built.

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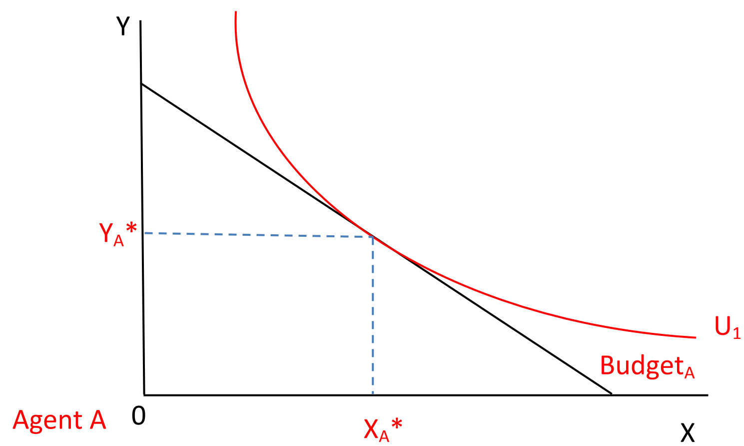
[3] Exogenous refers to something that has come from outside of a given system or an influence that does not stem from organic interactions within that system. Endogenous refers to those influences arising from within a system.

3. Do not transect one another
4. Are not tangent to one another
5. The consumer's level of utility is the same at any point on the curve
6. As the curve moves out and to the right the level of utility increases (and vice versa)

As with any other constrained optimization problem, we're looking for the maximum level of utility based on a given budget. If we assume that there are only two goods in an economy (X and Y), that the consumer is bound by a budget (M), and that the entire budget is to be expended through purchases/consumption of the goods, then the Utility Maximization Model and equation set looks something like this:

The consumer's budget can be represented as $P_x(X) + P_y(Y) = M$, which reads as the price of good X times the quantity of good X, plus the price of good Y times the quantity of good Y equals the budget (M).

We can also consider the consumer's utility function[4] which forms the indifference curve as $U = f(X, Y)$, which reads as utility is a function of the consumer's consumption of good X and good Y.



Finding the optimal levels of goods X and Y that allow for the highest level of consumer satisfaction then includes setting the budget equal to the utility, which is as easy and as difficult as it sounds. We'll look at two ways of doing this with using two different levels of math: 1) principles level algebra and 2) intermediate level calculus. We'll also use two different forms of utility equation to which will show very different tastes, preferences and optimal levels of X and Y.

Principles level algebra using marginal utilities

In this example we'll use a few pieces of information that must be given and then apply them to a set of rules or conditions that we've learned or developed. In this case the given information is simply arbitrarily selected, but in reality it would be the result of some analysis of a particular consumer and their budget. We're simply going to take it as given and remember that these are derived relationships or values and are not necessarily going to be the same for any two situations:

$P_x(X) + P_y(Y) = M = \$200$	This is simply the consumer's budget	(1)
$P_X = 1$ and $P_Y = 2$	These are the prices for two goods	(2)
$\frac{P_X}{P_Y} = 3\frac{Y}{X}$	A relationship between the two goods informing a particular form of consumer utility	(3)

[4] Utility functions are used by economists, but are very theoretical – that is, hard to expressly and specifically define. Their formation is most likely left up to sociologists, for which we economists are grateful.

We already addressed the consumer budget equation above so let's think a little about the issues involved in the indifference curve. We've identified the consumer's utility as $U = f(X, Y)$ and we can be assured that the consumer will choose the quantity of the goods they consumer based on a utility optimizing condition, or

rule, that tells us that the marginal utility of a good divided by the price of that good is equal to the marginal utility of the other good divided by the price of the other good. In this case we can write this as:

$$\frac{MU_X}{P_X} = \frac{MU_Y}{P_Y} \tag{4}$$

Which can be rewritten as

$$\frac{MU_X}{MU_Y} = \frac{P_X}{P_Y} \tag{5}$$

Since this is a utility maximizing condition, it's a target we want to achieve and as such governs much of the rest of what we'll do: remember that it simply tells us something about the marginal utilities of X and Y in respect to their prices. We have another equation (3) that also tells us something about X and Y in respect to their prices so let's rewrite these. Since we know the prices of these inputs we'll first substitute (2) into (3)

$$\frac{1}{2} = 3\frac{Y}{X} \tag{6}$$

Which can be rewritten as

$$X = \left(\frac{2}{1}\right)\left(\frac{3}{1}\right)Y = 6Y \quad \text{or} \quad X = 6Y \tag{7}$$

We'll think of this as one equation with two unknowns and to solve for the unknowns we need to have another equation with the same two unknowns present.

So let's use the budget equation given (1) to lead us to another equation that includes the two unknowns we're looking at

$$P_x(X) + P_y(Y) = \$200 \tag{8}$$

If we replace P_X and P_Y with the values given in (2) we can rewrite the equation as

$$X + 2Y = 200 \tag{9}$$

Now we have two equations (7) and (9) and two unknowns (X and Y) from which we're going to find the optimal values of X and Y this consumer will use to achieve maximum utility given a budget constraint.

We can substitute (7) into (9) such that

$$(6Y) + 2Y = 200 \quad \text{or} \quad 8Y = 200$$

From here we can rewrite the equation as

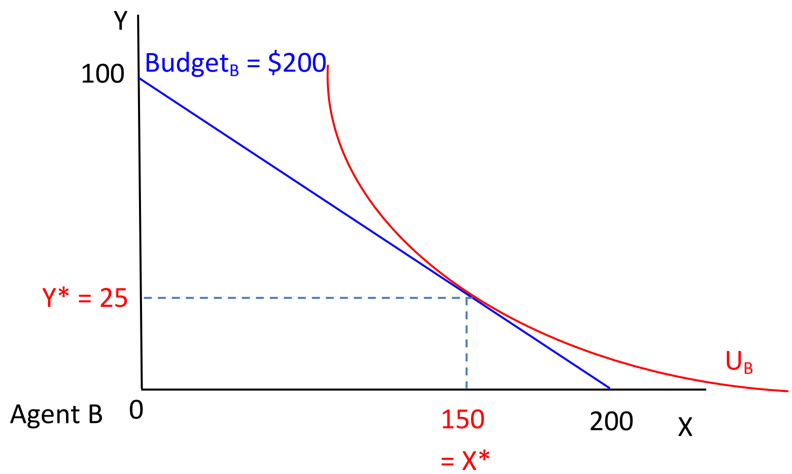
$$Y = \frac{200}{8} = 25 \quad \text{or} \quad Y^* = 25 \qquad (10)$$

Now we simply need to substitute (10) into (9) to get

$X + 2(25) = 200$ or with some simple algebra $X^* = 150$

So our optimal quantities of X * and Y * are 150 and 25 respectively.

We need to think about how this should look in our graphic model using the budget and indifference curves. You might recall that we obtain the intercepts of the respective axes by dividing the budget by the price of the input represented by the axis.



Intermediate level calculus using the Lagrangian method

If we assume a Cobb Douglas utility function with constant returns to scale[5] in the form of $U = f(X, Y) = X^{\alpha}Y^{1-\alpha}$. Note that this represents a very different set of tastes and preferences than the utility identified in the principles level example and the optimal X and Y will be different as well. Our goal is to maximize utility subject to some budget constraint, which can be written as:

$$Max\ U = U(X,Y)\ subject\ to\ M = f(P_i, X, Y) \ \text{or}\ Max\ U = X^{\alpha}Y^{1-\alpha} + \lambda[M - P_X(X) - P_Y(Y)] \qquad (1)$$

Recall the budget equation

$$P_x(X) + P_y(Y) = M \qquad (2)$$

This requires some simple calculus (partial derivatives) to find the solution to this problem, yields Marshallian demand equations for X and Y, and as a result can then reveal optimal values of X^* and Y^* as noted in the model.

[5] Cobb Douglas utility function with constant returns to scale simply refers to a particular equation form for which the sum of the exponents equals one (1)

To solve for X* and Y* we need to find the partial derivatives of the equation

$$\frac{\partial U}{\partial X} = aX^{a-1}Y^{1-a} - \lambda P_X \quad (3)$$

$$\frac{\partial U}{\partial Y} = (1-a)X^{a}Y^{1-a-1} - \lambda P_Y \quad (4)$$

$$\frac{\partial U}{\partial \lambda} = M - P_X X - P_Y Y \quad (5)$$

We can then set (3) and (4) each to zero, equate the two equations (since 0 = 0), and move the exogenous variables (P_X and P_Y) to the right side of the equations and then solve for X and separately for Y (these reveal the Marshallian demand equations):

$$\frac{aX^{a-1}Y^{1-a}}{(1-a)X^{a}Y^{-a}} = \frac{\lambda P_X}{\lambda P_Y} \quad (6)$$

$$\frac{a}{1-a}\frac{Y}{X} = \frac{P_X}{P_Y} \quad (7)$$

Solve for Y and separately for X to find the Marshallian demand equations

$$Y = \frac{1-a}{a}\frac{P_X}{P_Y}\,\mathrm{X} \quad (8)$$

$$X = \frac{a}{1-a}\frac{P_Y}{P_X}\,Y \quad (9)$$

Substitute (8) into (2) to find X*

$$P_x X + P_y\left(\frac{1-a}{a}\frac{P_X}{P_Y}X\right) = P_X X\frac{1-a}{a}P_X X = P_X X\left(\frac{1-a}{a}+1\right) = P_X X\left(\frac{1-a}{a}+\frac{a}{a}\right) = P_X X\left(\frac{1}{a}\right) = M \quad (10)$$

$$P_X X = aM \quad (11)$$

$$X^* = \frac{aM}{P_x} \quad (12)$$

Now substitute (12) into (2) to find Y*

$$P_x\frac{aM}{P_x} + P_y Y = aM + P_y Y = M \quad (13)$$

$$P_y Y = M - aM = M(1-a) \quad (14)$$

$$Y^* = \frac{M(1-a)}{P_Y} \quad (15)$$

So now that we have the form of X* and Y* we'd like to find their respective values, but to do this we need to identify values for P_X, P_Y, α and M. We consider that these are exogenously set so we'll arbitrarily assume that P_X = 1, P_Y = 2, α = .25 and M =200 and plug these into (12) and (15):

$$X^* = \frac{.25(200)}{1} = 50$$

$$Y^* = \frac{200(1-.25)}{2} = \frac{150}{2} = 75$$

With these values (including those we've assumed) we can better articulate the Utility Maximization Model including the X and Y intercepts of the budget curve. To do this we'll rewrite the budget equation with the appropriate values for P_X, P_Y, and M, so we'll substitute these into (2):

$$X + 2Y = 200 \qquad (16)$$

And then we can find the intercepts by setting X and Y to zero (0) in separate equations. This is a different method of finding the intercepts than the one we used in the principles level example, but it's no more or less accurate and has about the same level of difficulty.

$$0 + 2Y = 2Y = 200; \; Y = 100 \qquad (17)$$

$$X + (0)Y = X = 200 \qquad (18)$$

We need to think about how this should look in our graphic model using the budget and indifference curves.