Net Present Value (NPV) and Discounted Present Value Calculations[1]
Instructional Primer[2]

It's often helpful to understand how future streams of income (benefits) or even a lump sum payment relate to value in today's terms, what the current value (in real terms) is of a benefit offered in the future, or to estimate the future benefits (costs) of a transaction or series of transactions in order to decide whether or not to enter into the exchange. To do this we need to understand a series of variables and through the use of a Present Value (PV) equation we can specify a response.

Consider the college student who is trying to decide if the cost of an education is worthwhile or the business person trying to determine if the expected benefits of an investment will yield enough surplus to make it worth tying up capital. These are numeric exercises many have learned to do with complex calculators, but it's good to understand how they come together and something of the math behind them.

We'll look at two cases, consider the variables at play in them, and then work out the relevant equations for each.

Case One	Suppose Jessica, a 29 year old Utah native, chooses to return to college to earn a master's degree in economics under the belief that she can earn $25,000 more per year with the degree than she is earning now ($40,000 per year). Jessica is realistic about how she values opportunities and resources and has a personal discount rate of 4%. Jessica plans to take a Graduate Prep course up front at a cost of $5,000. The cost of getting a master's in economics is approximately $45,000 ($15,000 annually for each of three years) and most people can earn the degree while still employed. Jessica is very competitive and wants to earn straight A's and write a killer thesis so she has decided that she will cut back her hours at work by 40% - her pay will also reduce by the same percentage – so she's also committed to reducing her annual expenses by $,5000. She also knows that upon completion of college it will cost her more to live than it does now and she expects her expenses to increase by $2,500 annually. Jessica will pay for her schooling at the beginning of each year and wants to access the value of the experience ten years after she finishes school. We're not taking into account any inflation in this Case for simplicity's sake.
Case Two	Lauren wants to buy a used car for $12,000 and has convinced her father to lend her the money under the premise that she'll pay him back in one lump sum installment at the end of three years. She is offering him 3% on the loan. Lauren has the money and receives 1.5% on her savings, but if she uses it for the car she won't have any liquid reserves and she wants to know how much of her money she needs to set aside today to make the payment to her father in the future.
Case Three	Becca is planning on purchasing a piece of equipment for $25,000 and wants to know what its depreciated value will be after 7 years of use, assuming an annual depreciation rate of 6%.

[1] This primer is intended to present an abbreviated discussion of the included economic concepts and is not intended to be a full or complete representation of them or the underlying economic foundations from which they are built.

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The following sets out the variables, equations and methods necessary to determine how or whether to proceed with these transactions. In the Case One, Jessica needs to look at the costs of her education, including opportunity costs, and compare them against the present value of the future stream of benefits. In Case Two Lauren simply needs to know how much she needs to put away today to meet her obligations tomorrow.

Variable	**Description**
Time	The number of units of time – this may be in months, days, years, etc.
Present Value	This may be an ending value representing a cost (-) or a benefit (+)
Future Value	This may be a starting value representing a cost (-) or a benefit (+)
Rate	This is either a rate being charged, such as interest, or a personal discount rate that you would require to make it worth your while to enter into the subject transaction. It's important that the rate variable be reflected in the same increments as the time variable: for example, if we're considering a 5% annual rate, but are looking at 15 months, we need to a monthly rate (rate/12).
Payments	Payments can be thought of as a stream of exchange, some units of which may be positive (+) and some of which may be negative (-). If something is a cost then it would have a negative value, if it is a benefit, then it would have a positive value – it may be that for a given unit of time there are both costs and benefits that need to be accounted for; in this case it would be the net amount with its respective sign. Like the rate variable, it's important to make sure the payment variable is expressed in the same terms as the time variable; i.e. if you're looking at a monthly rent payment, but calculating an annual expression you would need to multiply the monthly amount by 12.

There are three basic structures when considering these cases, each of which has its own unique equation form:

Case	**Equation**
Case One: There are numerous payments (costs/benefits) under consideration; with at least one net value for each increment of time (it is possible for that amount to be 0). Each payment corresponds to a particular time increment. So if you're looking a 10-year duration, then there should be 10 separate increments of time.	$PV = \sum \frac{\beta^t}{(1+r)^t}$ This is the equation form to use for Case One and can be read as the Present Value (PV) is equal to the sum of the payments at time t, each one divided by 1 plus the rate (raised to the power of t). The time variable (t) is used both as an expressor and as an exponent.
Case Two: There is only one "payment" in this case, which becomes synonymous with the Future Value (FV) = 12,000 which we expect to be greater than the PV. We know that the time variable (t) = 3, and the rate variable (r) = 1.5%, which is the difference between the rate she will earn (1.5%) and the rate she has agreed to pay (3%).	$PV = \frac{\beta_t}{(1+r)^t}$ This is the equation form to use for Case Two and can be read as the present value is equal to the Future Value (FV) at time t, divided by 1 plus the rate (raised to the power of t). In this case the time variable (t) is used both as an expressor and as an exponent.

Case Three: There is no payment in this case and we're calculating the Future Value (FV), so we'll treat this as a case with no payments, but for which we know the Present Value (PV) = 25,000, which we expect to be greater than the FV.	$$PV = \frac{FV}{(1-r)^t}$$ This is the equation form to use for Case Three and can be read as the PV is equal to the FV at time t, divided by 1 minus the rate (raised to the power of t). Note that this is subtractive rather than additive representing depreciation's value reducing function.

Case One: Jessica's Education

Let's take this apart and look at the variables in detail.

Time	Present Value	Future value	Rate	Payments
13	To be determined	Undefined	4%	See table

Time: This Case plays out over a 13 time frame (3 years while Jessica is in school and another 10 years after she has finished her schooling).

Present Value: The present value (PV) is the variable we're going to calculate, but what might we be looking for here? Since the cost of Jessica's case is all built into the case, we're going to compare PV to 0 as follows:

- if PV > 0, then Jessica will obtain economic benefit from the case – the benefits will be greater than the costs
- if PV < 0, then the case will cost Jessica more than it benefits her
- if the PV = 0, then Jessica should be indifferent to whether or not she moves ahead with her plan – her costs will simply equal her benefits.

Future Value: With this particular equation form we need to know future value (FV) or stream of payments, but not both. By ignoring FV in this case, we presume there is no additional value unaccounted for at the end of the case.

Rate: The rate we're interested in here is Jessica's personal discount rate. This is an arbitrary value Jessica has chosen to help her assess the case. It is founded on her personal expectations and only tangentially connects to interest rates in the market. If this were an investment analysis or a debt calculation, we might have chosen a market rate for comparison.

Payments: The following identifies the net payments arising from Jessica's case. Note that these are not all positive and that negative values suggest that the net stream of payments represents outgo rather than income. Also note that Jessica has a $5,000 expense for tuition, etc. in year 0, this represents the expense of the Graduate Prep that Jessica must pay in advance. Assume that all other incomes and expenses are accounted for by Jessica at the beginning of the year expenses (this may not be all that realistic, but for illustrative purposes it simplifies the matter). Think about these values as follows:

- In year 0 Jessica pays $5,000 for the Graduate Prep Course – this is an upfront expense and occurs prior to anything else and before any time passes.
- In years 1-3 Jessica incurs $26,000 in net expenses. This represents $16,000 in lost earnings, $15, 000 in tuition, etc. and recognizes $5,000 in foregone living expenses.
- In years 4 -13 Jessica $22,500 in net benefits. This includes the $25,000 increase in her income less the $2,500 increase living expense she expects as a result of having improved her condition and taken a higher paying job.

End of Year	Expected earnings increase	Lost earnings while in school	Current cost of living decrease	Future cost of living increase	Tuition, books, etc.	**Total**
0					-5000	**-5000**
1		-16000	5000		-15000	**-26000**
2		-16000	5000		-15000	**-26000**
3		-16000	5000		-15000	**-26000**
4	25000			-2500		**22500**
5	25000			-2500		**22500**
6	25000			-2500		**22500**
7	25000			-2500		**22500**
8	25000			-2500		**22500**
9	25000			-2500		**22500**
10	25000			-2500		**22500**
11	25000			-2500		**22500**
12	25000			-2500		**22500**
13	25000			-2500		**22500**

With this all of this in mind, lets form the equation, first using the general form (1), then expanding to include the specified time variable (2), then replacing variables with values (3), and finally replacing terms with values (4) to reach the summation (5):

$$PV = \sum \frac{\beta_t}{(1+r)^t} \tag{1}$$

$$PV = \frac{\beta_0}{(1+r)^0} + \frac{\beta_1}{(1+r)^1} + \frac{\beta_2}{(1+r)^2} + \frac{\beta_3}{(1+r)^3} + \frac{\beta_4}{(1+r)^4} \ldots . \frac{\beta_{13}}{(1+r)^{13}} \tag{2}$$

The "…" notation in this equation simply indicates that term for years 5-12 is the same as the term for year 5 with the exception of the time (t) variable.

$$PV = -\frac{5000}{(1+.04)^0} - \frac{26000}{(1+.04)^1} - \frac{26000}{(1+.04)^2} - \frac{26000}{(1+.04)^3} + \frac{22500}{(1+.04)^4} \ldots . \frac{22500}{(1+.04)^{13}} \tag{3}$$

$$\begin{aligned} PV = & -5{,}000 - 25{,}000 - 24{,}038 - 23{,}114 + 19{,}233 + 18{,}493 + 17{,}782 \\ & +17{,}098 + 16{,}441 + 15{,}808 + 15{,}200 + 14{,}616 + 14{,}053 + 13{,}513 \end{aligned} \tag{4}$$

$$PV = 85{,}085 \tag{5}$$

Based on comparing PV against 0, it appears obvious that Jessica should participate in this education program – over a period of 13 years it will yield $85,085 in current value dollars more than it will cost her.

Case Two: Lauren's car payment

Time	Present Value	Future value	Rate	Payments
3	To be determined	12,000	1.5%	Undefined

Time: This Case plays out over a 3 time frame with Lauren's payment due to her father at the end of three years.

Present Value: The present value (PV) is the amount Lauren needs to put away to retire her debt in three years and it's the variable we're going to calculate. Rather than compare it to 0 as we did in Case One, we're going to consider it independently.

Future Value: The future value is the amount Lauren needs to retire the debt: $12,000.

Rate: The rate is this case is the difference between what Lauren expects to earn on her money and the rate she needs to pay her father. This is a simplistic way of looking at this, but yields relatively accurate results nonetheless. In this case the difference is 1.5% (3% - 1.5%).

Payments: This form of the equation doesn't include payments, we've used FV instead.

With this all of this in mind, lets form the equation, first using the general form (1), then expanding to include the specified time variable (2), then replacing variables with values (3), and finally replacing terms with values (4) to reach the summation (5):

$$PV = \frac{\beta_t}{(1+r)^t} \tag{6}$$

$$PV = \frac{\beta_3}{(1+r)^3} \tag{7}$$

$$PV = \frac{12000}{(1+.015)^3} \tag{8}$$

$$PV = \frac{12000}{(1.015)^3} \tag{9}$$

$$PV = 11476 \tag{10}$$

If you think about it, with this form of the equation you can easily see that we could have solved for any one of the relevant variables if all of the others had been present. For example, had we not know the FV, but knew the amount Lauren had to set aside to retire the debt (PV = $11,476), we could have solved for FV and seen that it is $12,000. We could have done the same thing with the rate and time variables – it becomes a matter of simple algebraic substitution. Because of the greater complexity of the equation we used in Case One, such substitution is feasible, but more complex.

Case Three: Becca's equipment purchase with expected depreciation

Time	Present Value	Future value	Rate	Payments
7	$25,000	To be determined	6%	Undefined

Time: This Case plays out over a 7 time frame with Becca seeking to determine the depreciated value at the end of 7 years.

Present Value: The present value (PV) is the amount Becca will spend on the equipment; $25,000.

Future Value: The future value is the amount Becca is trying to determine, so it's the value we'll calculate.

Rate: The rate is the depreciation rate, or 6%.

Payments: This form of the equation doesn't include payments, we're considering FV instead.

With this all of this in mind, lets form the equation, first using the general form (1), then expanding to include the specified time variable (2), then replacing variables with values (3), and finally replacing terms with values (4) to reach the summation (5):

$$PV = \frac{FV}{(1-r)^t} \tag{11}$$

$$PV = \frac{FV}{(1-r)^7} \tag{12}$$

$$25{,}000 = \frac{FV}{(1-.06)^7} \tag{13}$$

$$FV = 25{,}000 * .94^7 \tag{14}$$

$$FV = 16{,}212 \tag{15}$$

Taking Case Three a step further

There are numerous ways these equations can be combined. Let's think about Becca's decision for a moment. So far all we've done is calculated the depreciated value of the equipment she's about to purchase. What if she also wanted to compare that value with the expected present value of the equipment's production, in other words, the profit expected from using the equipment in a productive capacity? This becomes two different equations that we can then compare. For this we'll need to know the variables representing the economic outcome of the production process and we'll use an equation similar to the one we used to analyze Jessica's decision.

Let's assume that Becca plans to use the equipment to manufacture a canned food product line with expected profits of $1,000 annually for the same seven years being considered in the depreciation equation. Becca expects to compare the Net Present Value of both the earnings and the equipment purchase combined in an effort to decide if this is a good idea.

Becca's equation to assess the PV of the earnings stream is $PV = \sum \frac{\beta_t}{(1+r)^t}$, with t = 7. Becca has a different expected rate (4%) than the depreciation rate (6%), and we already know that the payment stream (β_t) is equal to $1,000 annually. Becca's equation then is articulated as follows:

$$PV = \sum \frac{1000_7}{1.04^7} \tag{16}$$

$$PV = \frac{1000_1}{1.04^1} + \frac{1000_2}{1.04^2} + \cdots + \frac{1000\beta_7}{1.04^7} \tag{17}$$

$$PV = 962 + 925 + 889 + 855 + 822 + 790 + 760 \tag{18}$$

$$PV = 6{,}003 \tag{19}$$

So now we have the present value of Becca's expected profits from production, but what do we compare this against? The figure is unambiguously positive, but a comparison against 0 (as in Case One) isn't relevant. We need to compare this against Becca's cost; not her production costs, those are endogenized (internalized) in the expected profits, but the cost of owning the equipment that allows her to enter production. This cost can be seen as the difference between the purchase price of the equipment and its depreciated value: $PV = \frac{FV}{(1-r)^t}$. We're presuming that the depreciated value is the same as the market value at the end of the even years.

We see that Becca paid $25,000 for the equipment and if she were to sell the equipment at its depreciated value of $16,212 (equation 15) she would recognize a cost of $8,788. In this case we see that the PV of Becca's expected profits of $6,003 (equation 19) is less than $8,788, so she would likely not choose to purchase the equipment and enter production.

Recall that the equation we used is really two equations set against each other as follows:

$$PV_{purchase} - \frac{FV}{(1-r_{depreciation})^t} = \sum \frac{1000}{(1+r_{production})^t} \tag{20}$$

Note that the rate (r) has been differentiated by the terms production and profit; this is because the Becca's analysis involved two separate and distinct rates. Had the situation employed only one rate we could have combined these equations differently. Also, you might note that we can rearrange the equation to compare PV against 0, as follows:

$$PV_{purchase} = \frac{FV}{(1-r_{depreciation})^t} + \sum \frac{1000}{(1+r_{production})^t} \quad (21)$$

Had we chosen to do this, we could say that we endogenized the cost (depreciation) into the equation, in which case we could then compare $PV_{purchase}$ against 0, with a positive value suggesting that the enterprise is warranted, a negative value suggesting it is not, and a value equal to 0 leaving us indifferent.